

ORTHOGONAL POLYNOMIALS ON THE SPHERE WITH OCTAHEDRAL SYMMETRY

BY

CHARLES F. DUNKL¹

ABSTRACT. For any finite reflection group G acting on \mathbf{R}^N there is a family of G -invariant measures ($h^2 d\omega$, where h is a certain product of linear functions whose zero-sets are the reflecting hyperplanes for G) on the unit sphere and an associated partial differential operator ($L_h f := \Delta(fh) - f\Delta h$; Δ is the Laplacian). Analogously to spherical harmonics, there is an orthogonal (with respect to $h^2 d\omega$) decomposition of homogeneous polynomials, that is, if p is of degree n then

$$p(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} |x|^{2j} p_{n-2j}(x),$$

where $L_h p_i = 0$ and $\deg p_i = i$ for each i , but with the restriction that p and p_i must all be G -invariant. The main topic is the hyperoctahedral group with

$$h(x) = (x_1 x_2 \cdots x_N)^\alpha \left(\prod_{i < j} (x_i^2 - x_j^2) \right)^\beta.$$

The special case $N = 2$ leads to Jacobi polynomials. A detailed study of the case $N = 3$ is made; an important result is the construction of a third-order differential operator that maps polynomials associated to h with indices (α, β) to those associated with $(\alpha + 2, \beta + 1)$.

Introduction. Regular figures have long been objects of fascination. We study here the interplay between them and the theory of orthogonal polynomials. This will also touch on finite groups generated by reflections and their polynomial invariants, as well as a generalization of spherical harmonics. The underlying situation throughout will consist of spaces of homogeneous polynomials on \mathbf{R}^N that satisfy a second-order differential equation and whose restrictions to the unit sphere have an orthogonality relation with respect to a weight function, and of a finite reflection group for which all these objects are invariant.

The weight (measure) $h^2(x) d\omega(x)$ on the unit sphere S in \mathbf{R}^3 , where

$$h(x) = (x_1 x_2 x_3)^\alpha ((x_1^2 - x_2^2)(x_1^2 - x_3^2)(x_2^2 - x_3^2))^\beta,$$

$\alpha, \beta = 1, 2, \dots$ ($d\omega$ is surface area), and the associated polynomials will be studied most intensely. Some of the highlights of the theory are as follows:

Define the differential operator

$$L_h f := \Delta(fh) - f\Delta h$$

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(for f smooth on \mathbf{R}^3 , and Δ is the Laplacian). Then any polynomial solution p of $L_h p = 0$ is invariant under the octahedral group and hence is a polynomial in e_1, e_2, e_3 where

$$e_1 := \sum_i x_i^2, \quad e_2 := \sum_{i < j} x_i^2 x_j^2, \quad e_3 := x_1^2 x_2^2 x_3^2.$$

Furthermore, if p and q are homogeneous polynomials of different degrees, $L_h p = 0$ and $L_h q = 0$, then $\int_S p q h^2 d\omega = 0$. The subset $\{(x_1, x_2, x_3) \in S: x_1 \geq x_2 \geq x_3 \geq 0\}$ corresponds to the region $E := \{(e_2, e_3): e_2, e_3, D \geq 0\}$ where $D = e_2^2 - 4e_3^3 - 4e_3 - 27e_3^2 + 18e_2e_3$ (the restriction of the discriminant), and the measure $h^2(x) d\omega(x)$ corresponds to $\frac{3}{\pi} e_3^{\alpha-1/2} D^{\beta-1/2} de_2 de_3$ on E .

In a sense we consider orthogonal polynomials on E , but there is a significant difference between this situation and the usual one. In the latter the x and y variables are each given degree 1, whereas here e_2 and e_3 are of degrees 2 and 3, respectively (in $\{x_i^2\}$), so that, for example, $e_2^3 + e_3^2$ is homogeneous of degree 6.

The statements about L_h and $h^2 d\omega$ actually can be applied to any finite reflection group on \mathbf{R}^N , and in §2 this general situation will be presented. The key observation is that h is a product of linear functions such that the reflections in their zero sets (hyperplanes) generate a finite reflection group under which h is (relatively) invariant. The relation of this to ordinary harmonic polynomials will also be considered.

In §3 the general theory will be applied to the function

$$h(x) = (x_1 x_2 \cdots x_N)^\alpha \cdot \left(\prod_{i < j} (x_i^2 - x_j^2) \right)^\beta$$

on \mathbf{R}^N ($\alpha, \beta = 1, 2, \dots$), and solutions of $L_h p = 0$ will be found in terms of e_i ($1 \leq i \leq N$), where e_i is the elementary symmetric function of degree i in $x_1^2, x_2^2, \dots, x_N^2$.

Finally a detailed investigation of this function h will be made in the cases $N = 2$ and $N = 3$. We present the $N = 2$ version, which is essentially the theory of Jacobi polynomials, by way of motivation for further studies. For $N = 3$ we describe a third-order differential operator which maps polynomials associated to the parameters (α, β) to those for $(\alpha + 2, \beta + 1)$ (analogously to the derivative of $P_n^{(\alpha, \beta)}$ being a multiple of $P_{n-1}^{(\alpha+1, \beta+1)}$, for Jacobi polynomials).

Notation and conventions: For an element $x = (x_1, x_2, \dots, x_N) \in \mathbf{R}^N$ the norm $|x| := (\sum_i x_i^2)^{1/2}$. The unit sphere S in \mathbf{R}^N is the set $\{x: |x| = 1\}$. The rotation-invariant surface measure $d\omega$ on S is the normalized one, that is, $\int_S d\omega = 1$.

All the functions will be real valued, while parameters such as α, β will be integer valued (extensions to real values of α, β are fairly obvious).

For integers m, n let $m \wedge n := \min(m, n)$, $m \vee n := \max(m, n)$. For a finite set Ω , $|\Omega|$ denotes the cardinality. The Laplacian on \mathbf{R}^N is $\Delta := \sum_{i=1}^N (\partial/\partial x_i)^2$. For differential operators D_1, D_2 the commutator is $[D_1, D_2] := D_1 D_2 - D_2 D_1$.

2. Reflection groups and the Laplacian. There is a close connection between polynomials on \mathbf{R}^N which are products of linear functions, the Laplacian, and finite reflection groups.

2.1. DEFINITION. A reflection group G is a finite subgroup of the orthogonal group on \mathbf{R}^N , some $N \geq 1$, that is generated by its reflections. A reflection is an involutory orthogonal transformation that has just one eigenvalue of -1 (with the rest being 1). Each reflection is associated to its hyperplane of fixed points, which in turn is associated to a linear function, its annihilator (determined up to a constant). For $n \geq 0$ let P_n^G be the space of homogeneous polynomials on \mathbf{R}^N of degree n that are invariant under G (that is, $p(\sigma x) = p(x)$ ($\sigma \in G, x \in \mathbf{R}^N$)).

We list the known results (see Stanley [11, pp. 485–488]) about reflection groups which are relevant to our study. There is a set of homogeneous invariant polynomials $\theta_1, \theta_2, \dots, \theta_N$ of degrees d_1, d_2, \dots, d_N such that the ring of G -invariant polynomials ($\sum_{n \geq 0} \oplus P_n^G$) is exactly the ring of all polynomials in $\theta_1, \dots, \theta_N$, and $|G| = \prod_{i=1}^N d_i$. Furthermore, the Jacobian $\partial(\theta_1, \dots, \theta_N)/\partial(x_1, \dots, x_N)$ is a scalar multiple of the product of the linear functions corresponding to the reflections in G (thus there are $\sum_{i=1}^N (d_i - 1)$ of these) (for a quick proof, see Steinberg [12]). A reflection group is said to be irreducible if it cannot be decomposed as a product of two reflection groups acting on nontrivial orthogonal subspaces of \mathbf{R}^N . The irreducible reflection groups are known (see Coxeter [2, Chapter 11]) and include the groups of type A_N (symmetric), B_N (hyperoctahedral), D_N , acting on \mathbf{R}^N , the dihedral groups on \mathbf{R}^2 , and a finite set of exceptional groups: E_6, E_7, E_8, F_4, H_3 (icosahedral), H_4 .

2.2. DEFINITION. Suppose h is a product of powers of linear functions on \mathbf{R}^N . We say h satisfies condition (*) for the (finite) reflection group G if the reflections corresponding to the linear functions in h generate G and $h(\sigma x) = \pm h(x)$ ($\sigma \in G$) (this property is called relative invariance).

Suppose the reflections w_j ($1 \leq j \leq m$) of the reflection group G are in the hyperplanes $\{x: l_j(x) = 0\}$, respectively (by convention, G fixes the origin so each l_j is homogeneous), and are partitioned into conjugacy classes C_i ($1 \leq i \leq M$). Then the most general h satisfying (*) for G is $\prod_{i=1}^M (\prod_{j \in C_i} l_j)^{\alpha_i}$ with each $\alpha_i \geq 1$. Note that if w is the reflection in $\{x: l(x) = 0\}$, then $\sigma^{-1}w\sigma$ is the reflection in $\{x: l(\sigma x) = 0\}$ ($\sigma \in G$).

Keeping in mind that we are concerned with weight functions on the sphere which are invariant under a reflection group, we now define a differential operator associated to such weights. The remarkable fact is that the polynomials in its kernel must themselves be invariant.

2.3. DEFINITION. For $h \in C^\infty(\mathbf{R}^N)$ the linear operator L_h is defined by $L_h f := \Delta(fh) - f\Delta h$ ($f \in C^\infty(\mathbf{R}^N)$).

2.4. THEOREM. Let l be a nontrivial linear function on \mathbf{R}^N and w the reflection in $\{x: l(x) = 0\}$. Suppose $h = l^\alpha g$, where α is a positive integer and g is a polynomial invariant under w and not divisible by l . If f is a nontrivial polynomial on \mathbf{R}^N such that $L_h f = 0$, then f is invariant under w and not divisible by l .

PROOF. Since Δ is invariant under orthogonal change of basis and translation we may assume $l(x) = x_N$, so by hypothesis $g(x)$ is even in x_N and not divisible by x_N . Let f be a polynomial solution of $L_h f = 0$ and write

$$f(x) = \sum_{m=0}^M p_m(x_1, \dots, x_{N-1}) x_N^m.$$

By symmetry the transformed function $\check{f}(x) = \sum_m p_m(-1)^m x_N^m$ also satisfies the equation, and so does the odd part $f_- = (f - \check{f})/2$. For the invariance we must show that $f_- = 0$. Suppose not, and let p_{2k+1} be the nonzero term of least index in f_- . Consider the expansion

$$\begin{aligned} \frac{1}{g(x)x_N^\alpha} L_h f_- = \sum_m \left\{ 2(2m+1)(m+\alpha) p_{2m+1} x_N^{2m-1} \right. \\ \left. + \left(\Delta p_{2m+1} + 2 \sum_{i=1}^{N-1} \frac{1}{g} \frac{\partial g}{\partial x_i} \frac{\partial}{\partial x_i} p_{2m+1} \right) x_N^{2m+1} \right. \\ \left. + 2(2m+1) \frac{1}{g} \frac{\partial g}{\partial x_N} p_{2m+1} x_N^{2m} \right\}. \end{aligned}$$

The nonzero term with the least exponent of x_N in this sum is

$$2(2k+1)(k+\alpha) p_{2k+1} x_N^{2k-1}$$

(x_N does not divide g so the denominators are no problem), implying that $L_h f_- \neq 0$, a contradiction.

Now let p_{2k} be the nonzero term of least index in f . The coefficient of x_N^{2k-2} in $(1/gx_N^\alpha)L_h f$ is $2k(2k-1+2\alpha)p_{2k}$, which can only vanish if $k=0$. \square

2.5. COROLLARY. Suppose h satisfies condition (*) for the reflection group G , and f is a polynomial satisfying $L_h f = 0$. Then f is invariant under G .

PROOF. Apply the theorem to each (maximal power of a) linear factor in h . \square

The following was found for \mathbf{R}^3 by Laporte [5]. Since this is not an easily accessible paper, we give a general proof.

2.6. COROLLARY. Let P be a product of (possibly inhomogeneous) linear functions on \mathbf{R}^N . Then $\Delta P = 0$ if and only if the reflections in the hyperplanes corresponding to the factors in P are exactly the set of reflections in a finite reflection group, and there are no repeated factors.

PROOF. By translation and rotation we may assume one of the factors is x_N . Write $P(x) = x_N f(x)$ and note that $L_h f = 0$ for $h(x) = x_N$. By the theorem f is invariant under the reflection in $\{x_N = 0\}$ and is not divisible by x_N (thus, no repeated factors). The invariance shows that the reflecting hyperplanes are mapped into each other by the action of any one of the reflections.

Let P_0 be the product of the homogeneous (degree 1) parts of the factors of P , that is, the part of highest degree of P . Since Δ is homogeneous, $\Delta P_0 = 0$. Assume the orthogonal complements of the hyperplanes in P_0 span \mathbf{R}^N , so the reflection group G

generated by the reflections in P_0 is finite because it acts on the finite set of orthogonal complements. Since G is a linear group, only the identity can fix all these subspaces.

Now the reflections in P are translations of the reflections of G . They must have a common fixed point, or else the group generated by the reflections in P would contain translations which produce infinitely many mirrors.

The converse is a well-known trick. Note that P changes sign under the action of a single reflection, and so does ΔP , but ΔP is of lower degree than P and so cannot be nontrivially divisible by each linear function corresponding to a reflection in the group; hence $\Delta P = 0$. \square

The orthogonality structure associated to L_h depends on Green's theorem.

2.7. LEMMA. *Let $h \in C^\infty(\mathbf{R}^N)$ and suppose f, g are homogeneous polynomials of different degrees and $L_h f = 0, L_h g = 0$. Then $\int_S fgh^2 d\omega = 0$.*

PROOF. By Green's second identity ($B := \{|x| < 1\}$)

$$\begin{aligned} \int_B (gL_h f - fL_h g)h dx &= \int_B \{hg\Delta(hf) - hf\Delta(hg)\} dx \\ &= c_N \int_S \left(g \frac{\partial}{\partial n} f - f \frac{\partial}{\partial n} g \right) h^2 d\omega \\ &= c_N (\deg f - \deg g) \int_S fgh^2 d\omega, \end{aligned}$$

where $\partial/\partial n$ is the normal derivative on S and c_N is the surface measure of S in \mathbf{R}^N . For homogeneous polynomials $\partial f/\partial n = (\deg f)f$ on S . \square

Any homogeneous polynomial p on \mathbf{R}^N has the unique expansion

$$p(x) = \sum_j |x|^{2j} p_{n-2j}(x),$$

where $n = \deg p$ and p_{n-2j} is harmonic and homogeneous of degree $n - 2j$ (thus any polynomial restricted to S is the boundary value of a harmonic polynomial). By Theorem 2.4 an analogous situation for the operator L_h can only exist for invariant polynomials.

2.8. DEFINITION. Let G be a finite reflection group and let h satisfy condition (*) for G . Then $H_n^h := \{p \in P_n^G : L_h p = 0\}$ (the G -invariant polynomials of degree n annihilated by L_h) ($n = 0, 1, 2, \dots$).

2.9. THEOREM. *If h satisfies (*) for G , then*

$$P_n^G = \sum_j \oplus |x|^{2j} H_{n-2j}^h \quad (n \geq 0).$$

PROOF. From Lemma 2.7 we have $H_n^h \cap |x|^{2j} H_{n-2j}^h = \{0\}$ for $j > 0$. This shows that P_n^G contains the indicated spaces as direct summands. The modified operator

$$\tilde{L}_h := \frac{1}{h} L_h = \Delta + \frac{2}{h} \sum_{i=1}^N \frac{\partial h}{\partial x_i} \frac{\partial}{\partial x_i}$$

maps P_n^G into P_{n-2}^G . To see this, assume, as in Theorem 2.4, that $h(x) = x_n g(x)$ with g even in x_N and not divisible by x_N . If $f \in P_n^G$ then f is even in x_N , and by the calculation in the proof of 2.4, $\tilde{L}_h f$ is an even polynomial in x_N with possibly rational coefficients, where the denominator is g . This argument is applied to every linear factor of h , and so $\tilde{L}_h f$ is a G -invariant polynomial. Clearly the degree of homogeneity of $\tilde{L}_h f$ is $n - 2$.

Hence $\dim P_{n-2}^G + \dim H_n^h \geq \dim P_n^G$. Assume the theorem is true for $n \leq k - 1$ (it is automatic for $k = 2$). Then $H_k^h \oplus |x|^2 P_{k-2}^G = \sum_j \oplus |x|^{2j} H_{n-2j}^h$ has dimension at least as large as P_k^G , hence is equal to P_k^G . By induction, the proof is finished. \square

Just as Prasad [9] (among others) gave an explicit expression for p_{n-2j} in the series $p(x) = \sum_j |x|^{2j} p_{n-2j}(x)$ with $p \in P_n$ and $\Delta p_{n-2j} = 0$, we can determine p_{n-2j} with $L_h p_{n-2j} = 0$, when p is G -invariant.

2.10. DEFINITION. For $n = 0, 1, 2, \dots$ define the operator

$$T_{h,n} p(x) := \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{|x|^{2i}}{4^i i! (-N/2 - \deg h - n + 2)_i} \tilde{L}_h^i p(x) \quad (p \in P_n^G)$$

(recall $\tilde{L}_h = (1/h)L_h$).

2.11. THEOREM. The operator $T_{h,n}$ is the orthogonal projection $P_n^G \rightarrow H_n^h$, and further, any $p \in P_n^G$ is expanded as

$$p(x) = \sum_{j=0}^{\lfloor n/2 \rfloor} |x|^{2j} p_{n-2j}(x)$$

with $p_{n-2j} \in H_{n-2j}^h$, where

$$p_{n-2j} := (4^j j! (N/2 + \deg h + n - 2j)_j)^{-1} T_{h,n-2j}(\tilde{L}_h^j p).$$

PROOF. From the identity

$$\Delta(|x|^{2m} f) = 4m(N/2 + m - 1 + \deg f) |x|^{2m-2} f + |x|^{2m} \Delta f,$$

for f homogeneous, follows

$$L_h(|x|^{2m} p) = 4m(N/2 + m - 1 + \deg h + \deg p) |x|^{2m-2} h p + |x|^{2m} L_h(p).$$

Let $p \in P_n^G$. Then by Theorem 2.9,

$$p = \sum_j |x|^{2j} p_{n-2j} \quad \text{with } p_{n-2j} \in H_{n-2j}^h.$$

By repeated use of the formula for L_h we obtain

$$\tilde{L}_h^m p = \sum_{i=m}^{\lfloor n/2 \rfloor} 4^m (-i)_m (i - N/2 - \deg h - n + 1)_m |x|^{2i-2m} p_{n-2i}.$$

It can now be directly verified that $T_{h,n} p = p_n$ (this uses the Chu-Vandermonde sum). (The form of $T_{h,n}$ was obtained after some calculations and judicious guessing.) Finally, note that p_{n-2m} is a multiple of the projection of $\tilde{L}_h^m p$ onto H_{n-2m}^h , namely $T_{h,n-2m} \tilde{L}_h^m p$. \square

The author thanks the referee for suggesting the use of powers of \tilde{L}_h to calculate the p_{n-2j} 's.

Koornwinder [4] used an operator essentially the same as L_h in describing two-variable analogues of Jacobi polynomials. The reflection groups he used either acted on the plane (\mathbf{R}^2) or acted reducibly on \mathbf{R}^3 .

3. The groups of type B_N and D_N . Here we will be concerned with the function

$$h(x) = (x_1 x_2 \cdots x_N)^\alpha \left(\prod_{i < j} (x_i^2 - x_j^2) \right)^\beta \quad (\alpha, \beta \geq 1).$$

This satisfies condition (*) for O_N , the (full) hyperoctahedral group on \mathbf{R}^N , which is the symmetry group of the cube $\{(\pm 1, \pm 1, \dots, \pm 1)\}$ and is generated by the reflections in $x_i = 0$, $x_i + x_j = 0$, $x_i - x_j = 0$ ($i, j = 1, \dots, N$ and $i \neq j$). The invariants for O_N are certainly even in each x_i , so for convenience we introduce variables $y_i := x_i^2$ ($1 \leq i \leq N$) and express $\tilde{L}_h = (1/h)L_h$ in terms of these.

3.1. PROPOSITION.

$$\begin{aligned} \tilde{L}_h &= \Delta + 2\alpha \sum_{i=1}^N \frac{1}{x_i} \frac{\partial}{\partial x_i} + 4\beta \sum_{i < j} \left(\frac{1}{x_i^2 - x_j^2} \right) \left(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j} \right), \\ \tilde{L}_h &= 4 \left(\sum_{i=1}^N y_i \left(\frac{\partial}{\partial y_i} \right)^2 + \left(\alpha + \frac{1}{2} \right) \sum_{i=1}^N \frac{\partial}{\partial y_i} + 2\beta \sum_{i < j} \left(\frac{1}{y_i - y_j} \right) \left(y_i \frac{\partial}{\partial y_i} - y_j \frac{\partial}{\partial y_j} \right) \right) \end{aligned}$$

in the x and y variables, respectively.

PROOF. This is a direct computation using

$$\tilde{L}_h = \Delta + \frac{2}{h} \sum_{i=1}^N \left(\frac{\partial h}{\partial x_i} \right) \frac{\partial}{\partial x_i}$$

and product formulas. \square

3.2. DEFINITION.

$$\begin{aligned} L &:= \sum_{i=1}^N y_i \left(\frac{\partial}{\partial y_i} \right)^2, \quad \delta_1 := \sum_{i=1}^N \frac{\partial}{\partial y_i}, \\ \delta_2 &:= \sum_{1 \leq i < j \leq N} \left(\frac{1}{y_i - y_j} \right) \left(y_i \frac{\partial}{\partial y_i} - y_j \frac{\partial}{\partial y_j} \right). \end{aligned}$$

Each of these differential operators lowers the degree of homogeneity (in y) by one. To indicate dependence on α, β we also define

$$L_{\alpha\beta} := L + \left(\alpha + \frac{1}{2} \right) \delta_1 + 2\beta \delta_2 = \frac{1}{4} \tilde{L}_h.$$

It is known that the invariants of O_N are the polynomials in the elementary symmetric functions in $\{x_i^2: 1 \leq i \leq N\}$, which we will denote by e_i , $0 \leq i \leq N$. They are defined by the generating function

$$\sum_{i=0}^N e_i t^i = \prod_{i=1}^N (1 + y_i t).$$

Further, for dealing with polynomials in the e_i , we will use the short forms $\partial_j := \partial/\partial e_j$.

3.3. PROPOSITION. For $1 \leq j \leq N$,

$$\delta_1 e_j = (N - j + 1) e_{j-1} \quad \text{and} \quad \delta_2 e_j = \binom{N-j+1}{2} e_{j-1};$$

equivalently,

$$\delta_1 = \sum_{j=1}^N (N - j + 1) e_{j-1} \partial_j, \quad \delta_2 = \sum_{j=1}^{N-1} \binom{N-j+1}{2} e_{j-1} \partial_j.$$

PROOF. We use a rather heavy-handed generating function approach since it gives neat proofs for these facts as well as for the next proposition. Let

$$F := \prod_{i=1}^N (t + y_i) = \sum_{i=0}^N e_i t^{N-i}.$$

Then

$$\begin{aligned} \delta_1 F &= \sum_{i=0}^N (\delta_1 e_i) t^{N-i} = \sum_{i=1}^N \prod_{j \neq i} (t + y_j) = \frac{\partial}{\partial t} F \\ &= \sum_{i=0}^{N-1} e_i (N - i) t^{N-i-1} = \sum_{i=1}^N (N - i + 1) e_{i-1} t^{N-i}. \end{aligned}$$

Also

$$\begin{aligned} \delta_2 F &= \sum_{i=0}^N (\delta_2 e_i) t^{N-i} \\ &= \sum_{1 \leq i < j \leq N} \left(\frac{1}{y_i - y_j} \right) \left(y_i \frac{\partial}{\partial y_i} - y_j \frac{\partial}{\partial y_j} \right) (t + y_i)(t + y_j) \prod_{k \neq i, j} (t + y_k) \\ &= t \sum_{i < j} \prod_{k \neq i, j} (t + y_k) = \frac{t}{2} \sum_{i=1}^N \sum_{j \neq i} \prod_{k \neq i, j} (t + y_k) \\ &= \frac{t}{2} \left(\frac{\partial}{\partial t} \right)^2 F = \sum_{i=1}^N \frac{(N - i + 1)(N - i)}{2} e_{i-1} t^{N-i}. \quad \square \end{aligned}$$

3.4. PROPOSITION. For polynomials in $\{e_j\}$,

$$L = \sum_{i, j=1}^N \left(\sum_{k=i \vee j}^{N \wedge (i+j-1)} (2k - i - j + 1) e_k e_{i+j-k-1} \right) \partial_i \partial_j.$$

PROOF. First observe that

$$Lf(e_1, \dots, e_N) = \sum_{i, j=1}^N \left(\sum_{k=1}^N y_k \frac{\partial e_i}{\partial y_k} \frac{\partial e_j}{\partial y_k} \right) \partial_i \partial_j f$$

(since $(\partial/\partial y_k)^2 e_i = 0$ for all i, k). The k -sum is the coefficient of $s^i t^j$ in the sum

$$\sum_{k=1}^N y_k s t \prod_{m \neq k} (1 + s y_m)(1 + t y_m),$$

which equals

$$\begin{aligned} st \sum_{k=1}^N \frac{y_k}{(1+sy_k)(1+ty_k)} \prod_{m=1}^N (1+sy_m)(1+ty_m) \\ = \frac{st}{t-s} \sum_{k=1}^N \left(\frac{1}{1+sy_k} - \frac{1}{1+ty_k} \right) \prod_{m=1}^N (1+sy_m)(1+ty_m) \\ = \frac{st}{t-s} \left(\sum_{m=0}^{N-1} (N-m)e_ms^m \sum_{n=0}^N e_nt^n - \sum_{m=0}^N e_ms^m \sum_{n=0}^{N-1} (N-n)e_nt^n \right) \end{aligned}$$

(in the expression in Proposition 3.3 for δ_1 , we replace t by $1/t$ and multiply by t^{N-1} to get

$$\sum_{k=1}^N \prod_{m \neq k} (1+y_mt) = \sum_{m=0}^{N-1} (N-m)e_mt^m).$$

The expression above becomes

$$\begin{aligned} \sum_{m,n=0}^N (N-m)e_me_nst \left(\frac{s^mt^n - s^nt^m}{t-s} \right) \\ = \sum_{0 \leq m < n \leq N} ((N-m) - (N-n))e_me_nst \left(\frac{s^mt^n - s^nt^m}{t-s} \right) \end{aligned}$$

by skew-symmetry, which in turn equals

$$\sum_{0 \leq m < n \leq N} e_me_n(n-m) \sum_{k=1}^{n-m} t^{n+1-k} s^{m+k}.$$

Now we sum over the terms with $n+1-k=i$, $m+k=j$. \square

We describe a basis for $P_n^{O_N}$ which is labelled by the conjugate of the usual labelling (see Macdonald [6, p. 13]). Let λ denote an N -part (or less) partition, that is, $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_N)$ with $\lambda_i \in \mathbb{Z}_+$ and $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_N \geq 0$. The weight of λ is $|\lambda| := \sum_i \lambda_i$.

3.5. DEFINITION. Let Ω^N denote the set of N -part partitions, and for each $n = 0, 1, 2, \dots$ let

$$\Omega_n^N := \{\lambda \in \Omega^N : |\lambda| = n\} \quad \text{and} \quad \partial\Omega_n^N := \{\lambda \in \Omega_n^N : \lambda_1 = \lambda_2\}.$$

For $\lambda \in \Omega_n^N$, define the symmetric function

$$\phi(\lambda) := e_1^{\lambda_1 - \lambda_2} e_2^{\lambda_2 - \lambda_3} \dots e_N^{\lambda_N}.$$

For $n = 0, 1, 2, \dots$, $\alpha, \beta \geq 0$, let

$$H_n^N(\alpha, \beta) := P_{2n}^{O_N} \cap \ker L_{\alpha\beta} \quad (n \text{ is the } y\text{-degree}).$$

Note that $\deg \phi(\lambda) = 2|\lambda|$ (in the x -variables) and $\{\phi(\lambda) : \lambda \in \Omega_n^N\}$ is a basis for $P_{2n}^{O_N}$. Also $\dim P_{2n}^{O_N} = |\Omega_n^N|$, which is the coefficient of t^{2n} in $\prod_{i=1}^N (1-t^{2i})^{-1}$ (the Molien series for O_N).

3.6. PROPOSITION. For each $\alpha, \beta \geq 0$,

$$P_{2n}^{O_N} = \sum_{j=0}^n \oplus e^j H_{n-j}^N(\alpha, \beta).$$

Any continuous O_N -invariant function on S can be uniformly approximated by polynomials in $\ker L_{\alpha\beta}$.

PROOF. Theorem 2.9 applied to this situation gives the first assertion. Now suppose f is continuous and O_N -invariant on S and $\varepsilon > 0$. Then there is a polynomial p such that $|f - p| < \varepsilon$ on S . We may assume p is O_N -invariant (or else we use the projection: sum over all translates of p by elements of O_N , and divide by $|O_N|$). Now we expand p by degree of homogeneity and use the first part of the proposition (note that $e_1 = 1$ on S). \square

We claim that $\dim H_n^N(\alpha, \beta) = |\partial\Omega_n^N|$. Indeed (by Theorem 2.9)

$$\dim H_n^N(\alpha, \beta) = \dim(P_{2n}^{O_N}) - \dim(P_{2n-2}^{O_N}) = |\Omega_n^N| - |\Omega_{n-1}^N|,$$

and the correspondence $\lambda \rightarrow (\lambda_1 + 1, \lambda_2, \dots, \lambda_N)$ maps Ω_{n-1}^N into Ω_n^N , with the complement of the image being $\partial\Omega_n^N$.

This assertion will be explicitly demonstrated by descriptions of elements $\sum_{\lambda \in \Omega_n^N} c(\lambda) \phi(\lambda)$ of $H_n^N(\alpha, \beta)$ in which the coefficients $c(\lambda)$, $\lambda \in \partial\Omega_n^N$, can be arbitrarily chosen, determining the other coefficients (this explains our use of the boundary symbol ∂). To describe $L_{\alpha\beta} \phi(\lambda)$ we introduce operators on N -tuples λ of integers which change the weight by ± 1 . For $1 \leq i \leq j \leq N$ and $0 \leq k \leq (N - j) \wedge (i - 1)$, define $U(i, j, k)$ by

$$(U(i, j, k)\lambda)_m = \begin{cases} \lambda_m + 1 & \text{if } i - k \leq m \leq i, \\ \lambda_m - 1 & \text{if } j + 1 \leq m \leq j + k, \\ \lambda_m & \text{otherwise.} \end{cases}$$

Note that each $U(i, j, k)$ is invertible, $|U(i, j, k)\lambda| = |\lambda| + 1$, and $U(i, j, 0) = U(i, i, 0)$ for all $j \geq i$. Further, for $\lambda \in \Omega^N$, $U(i, j, k)^{-1}\lambda \in \Omega^N$ exactly when $(\lambda_i - \lambda_{i+1})(\lambda_j - \lambda_{j+1}) > 0$ for $k > 0$, or $(\lambda_i - \lambda_{i+1}) > 0$ for $k = 0$.

3.7. PROPOSITION.

$$\begin{aligned} L\phi(\lambda) = & \sum_{i=2}^N (\lambda_i - \lambda_{i+1})(\lambda_i - \lambda_{i+1} - 1) \sum_{k=1}^{(N-i) \wedge (i-1)} (2k+1) \phi(U(i, i, k)^{-1}\lambda) \\ & + 2 \sum_{2 \leq i < j \leq N} (\lambda_i - \lambda_{i+1})(\lambda_j - \lambda_{j+1}) \\ & \sum_{k=1}^{(N-j) \wedge (i-1)} (2k + j - i + 1) \phi(U(i, j, k)^{-1}\lambda) \\ & + 2 \sum_{i=1}^N (\lambda_i - \lambda_{i+1}) \left(2 \sum_{j=i}^N \lambda_j - (\lambda_i - \lambda_{i+1}) - 1 \right) \phi(U(i, i, 0)^{-1}\lambda). \end{aligned}$$

Also,

$$\delta_1 \phi(\lambda) = \sum_{i=1}^N (N - i + 1)(\lambda_i - \lambda_{i+1}) \phi(U(i, i, 0)^{-1}\lambda),$$

and

$$\delta_2\phi(\lambda) = \sum_{i=1}^{N-1} \binom{N-i+1}{2} (\lambda_i - \lambda_{i+1}) \phi(U(i, i, 0)^{-1}\lambda).$$

In these formulas, $\lambda_{N+1} = 0$.

PROOF. This is the result of applying Proposition 3.4 to $\phi(\lambda)$ and a change of index (replace k by $k - j$) in the summation there. The third term in $L\phi(\lambda)$ (corresponding to $k = 0$) is obtained by a telescoping summation over $j \geq i$ of the coefficients of $\phi(U(i, j, 0)^{-1}\lambda)$. \square

3.8. PROPOSITION. Let $\mu \in \Omega_{n-1}^N$ ($n \geq 1$). Then the coefficient of $\phi(\mu)$ in $L_{\alpha\beta} \sum_{|\lambda|=n} c(\lambda) \phi(\lambda)$ is

(3.1)

$$\begin{aligned} & \sum_{i=1}^N (\mu_i - \mu_{i+1} + 1) \left(2 \sum_{j=i}^N \mu_j - (\mu_i - \mu_{i+1}) + (N - i + 1) \right. \\ & \quad \cdot \left. \left(\alpha + \frac{1}{2} + (N - i)\beta \right) \right) c(U(i, i, 0)\mu) \\ & + \sum_{i=2}^{N-1} (\mu_i - \mu_{i+1} + 2)(\mu_i - \mu_{i+1} + 1) \sum_{k=1}^{(N-i) \wedge (i-1)} (2k + 1) c(U(i, i, k)\mu) \\ & + \sum_{2 \leq i < j \leq N-1} (\mu_i - \mu_{i+1} + 1)(\mu_j - \mu_{j+1} + 1) \\ & \quad \cdot \sum_{k=1}^{(N-j) \wedge (i-1)} (2k + j - i + 1) c(U(i, j, k)\mu) \end{aligned}$$

(the sums being extended over those (i, j, k) values for which $U(i, j, k)\mu \in \Omega^N$).

PROOF. We use 3.7 and in each term replace λ by the appropriate $U(i, j, k)^{-1}\mu$. \square

We show how the equations for $c(\lambda)$ resulting from $L_{\alpha\beta} \sum_{|\lambda|=n} c(\lambda) \phi(\lambda) = 0$ can be solved by an inductive process. The index for this is the degree of e_1 , so we define $r(\lambda) := \lambda_1 - \lambda_2$.

3.9. THEOREM. For any choice of values of $c(\mu)$ ($\mu \in \partial\Omega_n^N$) there is a unique element $\sum_{\lambda \in \Omega_n^N} c(\lambda) \phi(\lambda)$ of $H_n^N(\alpha, \beta)$, where each $c(\lambda)$ is a linear function in $\{c(\mu): \mu \in \partial\Omega_n^N\}$ with the coefficients being rational functions of α and β whose denominators are divisors of $r(\lambda)!(-2n + 2 - N(\alpha + \frac{1}{2} + (N - 1)\beta))_{r(\lambda)}$ (the shifted factorial). The expression for $c(\lambda)$ ($r(\lambda) \geq 1$) is recursively given as an explicit dependence on $c(\mu)$ with $r(\mu) = r(\lambda) - 1$ or $r(\lambda) - 2$.

PROOF. Fix $\lambda \in \Omega_n^N$ with $r(\lambda) = 1$ and let $\mu = U(1, 1, 0)^{-1}\lambda$ (that is, $\mu_1 = \lambda_1 - 1$, $\mu_i = \lambda_i$ for $i \geq 2$). To get $\sum_{|\mu|=n} c(\mu) \phi(\mu) \in H_n^N$ we set (3.1) equal to zero. The

coefficient of $c(\lambda) = c(U(1, 1, 0)\mu)$ is nonzero, indeed it is

$$r(\lambda) \left(2n - 1 - r(\lambda) + N \left(\alpha + \frac{1}{2} + (N - 1)\beta \right) \right).$$

The other $c(\mu)$ which appear in the equation have $r(\mu) < r(\lambda)$, indeed $r(U(i, j, k)\mu) = r(\lambda) - 1$ for $2 \leq i \leq j$ and $k \leq (N - j) \wedge (i - 1)$ unless $2 \leq i \leq j \leq N - i + 2$ and $k = i - 2$, in which case $r(U(i, j, i - 2)\mu) = r(\lambda) - 2$. \square

Admittedly we do not give closed-form expressions for the coefficients $c(\lambda)$, yet they are algorithmically defined. Further, a series $\sum_{|\lambda|=n} c(\lambda) \phi(\lambda)$ in $H_n^N(\alpha, \beta)$ is a generalization of a hypergeometric series (in fact, the ordinary ${}_2F_1$ -series appears for $N = 2$), and perhaps there are some summation formulas (which would be values of the functions at the points $(1, 1, \dots, 1)$, $(1, \dots, 1, 0), \dots, (1, 0, \dots, 0)$ in y -coordinates).

Finally, we discuss the weights associated to the groups of type D_N generated by the reflections in $\prod_{i < j} (x_i^2 - x_j^2)$, that is, $\alpha = 0$ and $\beta \geq 1$. It is known that the invariants for D_N are generated by e_1, \dots, e_{N-1} (the elementary symmetric functions in $\{x_i^2: 1 \leq i \leq N\}$ as before) and $f_N = x_1 x_2 \cdots x_N$. Note that any invariant polynomial can be uniquely written in the form $p_0(e_1, \dots, e_N) + f_N p_1(e_1, \dots, e_N)$.

3.10. THEOREM. For $h = (\prod_{i < j} (x_i^2 - x_j^2))^\beta$ and p a polynomial on \mathbf{R}^N , $L_h p = 0$ if and only if $p(x) = p_0(e_1, \dots, e_N) + f_N p_1(e_1, \dots, e_N)$ with $L_{0,\beta} p_0 = 0$ and $L_{1,\beta} p_1 = 0$.

PROOF. From Corollary 2.5 we know that for $L_h p = 0$ it is necessary that p be invariant under the reflection group (reflections in $x_i - x_j = 0$, $x_i + x_j = 0$, $i \neq j$). Thus $p = p_0 + f_N p_1$. By applying the reflection in $x_1 = 0$ to the equation $L_h p = 0$, we separate the two parts and find that $L_h p_0 = 0$ and $L_h(f_N p_1) = 0$. Clearly $L_h p_0 = 0$ if and only if $L_{0,\beta} p_0 = 0$. On the other hand, $L_h(f_N p_1) = L_{h f_N} p_1 + p_1(L_h f_N)$ (a simple calculation), and

$$\frac{1}{h} L_h f_N = \Delta f_N + 4\beta \sum_{i < j} \left(\frac{1}{x_i^2 - x_j^2} \right) \left(x_i \frac{\partial}{\partial x_i} - x_j \frac{\partial}{\partial x_j} \right) f_N = 0.$$

Thus $L_h(f_N p_1) = 0$ if and only if $L_{h f_N} p_1 = 0$ if and only if $L_{1,\beta} p_1 = 0$. \square

The group corresponding to $\alpha > 0$, $\beta = 0$ is reducible (it is Z_2^N); the invariants are arbitrary polynomials in x_1^2, \dots, x_N^2 and fall outside the scope of our study.

4. The special cases $N = 2$ and $N = 3$. In this section we suppress the " N " label on $H_n^N(\alpha, \beta)$ and Ω_n^N since it will be obvious from the context. We discuss the $N = 2$ case to show how Jacobi polynomials appear in the theory and as motivation for the search for a differential operator relating adjacent (α, β) values (for $\ker L_{\alpha\beta}$).

We specialize Proposition 3.8 to $N = 2$ and find the coefficient of $\phi(\mu_1, \mu_2)$ in $L_{\alpha\beta} \Sigma \{c(\lambda_1, \lambda_2) \phi(\lambda_1, \lambda_2): \lambda_1 + \lambda_2 = n, \lambda_1 \geq \lambda_2 \geq 0\}$ to be

$$(\mu_1 - \mu_2 + 1)(\mu_1 + 3\mu_2 + 2\alpha + 2\beta + 1)c(\mu_1 + 1, \mu_2) \\ + (\mu_2 + 1)(\mu_2 + \alpha + \tfrac{1}{2})c(\mu_1, \mu_2 + 1).$$

By Theorem 3.9, $\partial\Omega_n \neq \emptyset$ is necessary for $H_n(\alpha, \beta) \neq \{0\}$, that is, n must be even, say $n = 2m$, and then $c(m, m)$ can be arbitrary. Set $c(m, m) = 1$, $\mu_1 = m + j$, $\mu_2 = m - j - 1$ and obtain the recurrence

$$4(j+1)(2m+\alpha+\beta-j-1)c(m+j+1, m-j-1) + (m-j)(m-j+\alpha-\frac{1}{2})c(m+j, m-j) = 0$$

with the solution

$$c(m+j, m-j) = \frac{(-m)_j \left(-m-\alpha+\frac{1}{2}\right)_j}{j! (1-2m-\alpha-\beta)_j} \left(\frac{1}{4}\right)^j.$$

Thus $H_{2m}(\alpha, \beta)$ is spanned by the polynomial

$$e_2^m \cdot {}_2F_1\left(-m, \frac{1}{2} - m - \alpha; 1 - 2m - \alpha - \beta; e_1^2/4e_2\right),$$

which in turn (by reversing the order of summation) equals

$$\left(-\frac{1}{4}\right)^m \frac{\left(\alpha+\frac{1}{2}\right)_m}{(m+\alpha+\beta)_m} e_1^{2m} \cdot {}_2F_1\left(-m, m+\alpha+\beta; \alpha+\frac{1}{2}; 4e_2/e_1^2\right).$$

The latter is a scalar multiple of $e_1^{2m} P_m^{(\alpha-1/2, \beta-1/2)}(1-8e_2/e_1^2)$ (see Szegő [13, p. 62]). The orthogonality structure on the circle is for the weight

$$(x_1 x_2)^{2\alpha} (x_1^2 - x_2^2)^{2\beta} d\omega(x).$$

We express this in e -terms as a simple application of these ideas before we treat the harder case $N = 3$.

In terms of Jacobians,

$$dx_1 dx_2 = (\partial(e_1, e_2)/\partial(x_1, x_2))^{-1} de_1 de_2$$

in a restricted region of x values, for example $\{x_1 > x_2 > 0\}$, which corresponds to $0 < e_2 < e_1^2/4$. The unit circle corresponds to $\{e_1 = 1\}$ so the weight $h^2 d\omega$ is transformed to (up to a constant multiplier) $e_2^{\alpha-1/2} (1-4e_2)^{\beta-1/2} de_2$ (note that the discriminant $(y_1 - y_2)^2 = e_1^2 - 4e_2$), which is of course the appropriate beta-distribution for the Jacobi polynomials above.

4.1. *A differential relation.* The identity

$$\frac{d}{dt} P_m^{(\alpha-1/2, \beta-1/2)}(t) = \frac{m+\alpha+\beta}{2} P_{m-1}^{(\alpha+1/2, \beta+1/2)}(t)$$

(see Szegő [13, p. 63]) suggests that there is a differential operator in the e -variables which maps $H_{2m}(\alpha, \beta)$ to $H_{2m-2}(\alpha+1, \beta+1)$. This is indeed true and can be interpreted in terms of certain commutators. For $N = 2$ we have

$$L = e_1 \partial_1^2 + 4e_2 \partial_1 \partial_2 + e_1 e_2 \partial_2^2, \quad \delta_1 = 2\partial_1 + e_1 \partial_2, \quad \delta_2 = \partial_1$$

(see Proposition 3.3). Further, the commutator $[\delta_1, \delta_2] = -\partial_2$, which commutes with δ_1 and δ_2 , and

$$[L, \partial_2] = [L_{\alpha\beta}, \partial_2] = -4\partial_1 \partial_2 - e_1 \partial_2^2 = -(\delta_1 + 2\delta_2) \partial_2.$$

This identity is equivalent to

$$L_{\alpha+1,\beta+1}\partial_2 = (L_{\alpha\beta} + \delta_1 + 2\delta_2)\partial_2 = \partial_2 L_{\alpha\beta}.$$

This shows that ∂_2 maps $\ker L_{\alpha\beta}$ into $\ker L_{\alpha+1,\beta+1}$ while lowering the degree by 2. We will look for a similar situation in the case $N = 3$.

The case $N = 3$. The operators L, δ_1, δ_2 are now given by (see Propositions 3.3 and 3.4)

$$L = e_1\partial_1^2 + (e_1e_2 + 3e_3)\partial_2^2 + e_2e_3\partial_3^2 + 4e_2\partial_1\partial_2 + 6e_3\partial_1\partial_3 + 4e_1e_3\partial_2\partial_3;$$

$$\delta_1 = 3\partial_1 + 2e_1\partial_2 + e_2\partial_3; \quad \delta_2 = 3\partial_1 + e_1\partial_2.$$

We will use just (λ_2, λ_3) to describe an element $\lambda \in \Omega_n$, so Ω_n will be identified with $\{(\lambda_2, \lambda_3) \in \mathbb{Z}^2: \lambda_2 \geq \lambda_3 \geq 0, 2\lambda_2 + \lambda_3 \leq n\}$ and $\partial\Omega_n = \{(\lambda_2, \lambda_3) \in \Omega_n: \lambda_3 = n - 2\lambda_2\}$. We use the function $\mu(n) := |\partial\Omega_n|$ (as in Definition 1.2 of [3]). Observe that $\mu(n) = |\{m \in \mathbb{Z}: n/3 \leq m \leq n/2\}|$ and has the values $\mu(1) = 0$, $\mu(j) = 1$ for $j = 0, 2, 3, 4, 5$, and $\mu(n+6) = \mu(n) + 1$ for $n \geq 0$. The latter relation is the most obvious recurrence relation among $\{\dim H_n(\alpha, \beta)\}$. We will exploit this later.

4.2. PROPOSITION. For $n = 2, 3, \dots$, elements $\sum_{\lambda} c(\lambda_2, \lambda_3)\phi(\lambda_2, \lambda_3)$ of $H_n(\alpha, \beta)$ are obtained by arbitrarily specifying the values $c(\lambda_2, \lambda_3)$ for $\lambda_3 = n - 2\lambda_2$ (that is, $(\lambda_2, \lambda_3) \in \partial\Omega_n$) and then using the recurrence (in the order $\lambda_1 - \lambda_2 = n - 2\lambda_2 - \lambda_3 = r(\lambda) = 1, 2, \dots, n$)

$$c(\lambda_2, \lambda_3) = \left(1 / \left(r(\lambda) \left(r(\lambda) - 2n - \frac{1}{2} - 3\alpha - 6\beta \right) \right) \right) \\ \cdot \left\{ (\lambda_2 - \lambda_3 + 1)(\lambda_2 + 3\lambda_3 + 2\alpha + 2\beta + 1)c(\lambda_2 + 1, \lambda_3) \right. \\ \left. + (\lambda_3 + 1) \left(\lambda_3 + \alpha + \frac{1}{2} \right) c(\lambda_2, \lambda_3 + 1) \right. \\ \left. + 3(\lambda_2 - \lambda_3 + 2)(\lambda_2 - \lambda_3 + 1)c(\lambda_2 + 1, \lambda_3 - 1) \right\},$$

where terms of index outside Ω_n are omitted.

PROOF. This is the special case of 3.8 and 3.9 for $N = 3$. Note that $c(\lambda_2, \lambda_3)$ is a rational function of α and β with the denominator being a divisor of

$$r(\lambda)! \left(\frac{1}{2} - 2n - 3\alpha - 6\beta \right)_{r(\lambda)}. \quad \square$$

4.3. DEFINITION. For $\lambda_2 \geq \lambda_3 \geq 0$ let $F_{\lambda_2, \lambda_3}^{(\alpha, \beta)}$ be the (unique) element of $H_{2\lambda_2 + \lambda_3}(\alpha, \beta)$ that has $c(\lambda_2, \lambda_2, \lambda_3) = 1$ and $c(\lambda') = 0$ for all $\lambda' \in \partial\Omega$ with $\lambda' \neq \lambda$ (possible by Proposition 4.2). These functions form a basis for $\ker L_{\alpha\beta}$.

4.4. Special cases for α, β . From Corollary 2.6 we know that

$$h(x) = (x_1 x_2 x_3)^\alpha \left(\prod_{1 \leq i < j \leq 3} (x_i^2 - x_j^2) \right)^\beta$$

is harmonic exactly when α, β take the values 0 or 1 (4 cases). Thus for any $n \geq 0$, each of

$$H_n(0, 0), \quad (x_1 x_2 x_3) \cdot H_n(1, 0), \quad \prod_{i < j} (x_i^2 - x_j^2) \cdot H_n(0, 1), \\ \left(x_1 x_2 x_3 \prod_{i < j} (x_i^2 - x_j^2) \right) \cdot H_n(1, 1)$$

is a space of ordinary harmonic homogeneous polynomials (of degrees $2n, 2n + 3, 2n + 6, 2n + 9$, respectively) which are absolutely or relatively (multiplied by ± 1) invariant under the octahedral group O_3 (they are called “cubic harmonics”). These spaces were studied in [3], but with different bases for the invariants, namely the monomial basis (m_λ in Macdonald [6, p. 11]) and a basis related to S -functions [6, p. 23]

Because these polynomials are actually harmonic, there are orthogonal projections from the space of all harmonic polynomials onto these subspaces, and these projections can be represented by matrices of Krawtchouk polynomial values.

The weight function in (e_2, e_3) -space. We require the discriminant

$$D := \prod_{i < j} (x_i^2 - x_j^2)^2.$$

Since D is invariant under O_3 it must be a polynomial in e_1, e_2, e_3 , indeed

$$D = e_1^2 e_2^2 - 4e_2^3 - 4e_1^3 e_3 - 27e_3^2 + 18e_1 e_2 e_3$$

(see van der Waerden [14, p. 102]).

4.5. PROPOSITION. *The image of the mapping $(x_1, x_2, x_3) \rightarrow (e_1, e_2, e_3)$ of $\mathbf{R}^3 \rightarrow \mathbf{R}^3$ is exactly the set $\{(e_1, e_2, e_3): e_1, e_2, e_3, D \geq 0\}$.*

PROOF. The image is obviously contained in this set. Conversely, the cubic polynomial $(t + y_1)(t + y_2)(t + y_3) = t^3 + e_1 t^2 + e_2 t + e_3$ cannot vanish at positive values of t but must have three real roots $-y_1, -y_2, -y_3$ when $D \geq 0$ (see van der Waerden [14, p. 194]), hence each $y_i \geq 0$ and $x_i = \pm \sqrt{y_i}$ is real. \square

We will consider the polynomials in $H_n(\alpha, \beta)$ as functions on the domain

$$E := \{(e_2, e_3): e_2, e_3 \geq 0, D(1, e_2, e_3) \geq 0\}$$

(the variable e_1 , while always having the value 1, will be kept for purposes of homogeneity). For fixed e_2 , D is a quadratic polynomial in e_3 allowing the following explicit description of E :

$$E = \left\{ (e_2, e_3): 0 \leq e_2 \leq \frac{1}{3}, \max(0, 9e_2 - 2 - 2(1 - 3e_2)^{3/2}) \right. \\ \left. \leq 27e_3 \leq 9e_2 - 2 + 2(1 - 3e_2)^{3/2} \right\}.$$

From this we find that E is a three-sided figure with the following boundary: a line segment from $(0, 0)$ to $(\frac{1}{4}, 0)$, then a curve (part of $D = 0$) starting with slope $\frac{1}{2}$ leading to the cusp $(\frac{1}{3}, \frac{1}{27})$ (the slope is $\frac{1}{3}$ there), and the third side is the other part of $D = 0$, joining $(0, 0)$ to $(\frac{1}{3}, \frac{1}{27})$ and tangent to the other two sides at these points.

To change variables in the integrals we prefer to use Cartesian coordinates, rather than spherical polar coordinates, so as not to conceal the important Jacobian $\partial(e_1, e_2, e_3)/\partial(x_1, x_2, x_3)$.

4.6. LEMMA. Let p be a polynomial on \mathbf{R}^3 , homogeneous of degree $2k$. Then

$$\int_{\mathbf{R}^3} p(x) e^{-|x|^2} dx = \left(\pi^{3/2} \left(\frac{3}{2} \right)_k \right) \int_S p d\omega.$$

PROOF. In polar coordinates $x = r\omega$, where $\omega \in S$ and $r > 0$, so we have $dx = 4\pi r^2 dr d\omega$ (recall $d\omega$ is normalized). Then $p(r\omega) = r^{2k} p(\omega)$ and the r -integral is

$$\frac{1}{2} \Gamma(k + \frac{3}{2}) = \frac{1}{2} \Gamma(\frac{3}{2}) (\frac{3}{2})_k = \frac{1}{4} \Gamma(1/2) (\frac{3}{2})_k. \quad \square$$

4.7. PROPOSITION. Let $f(e_1, e_2, e_3)$ be a polynomial (considered as a function in x -variables). Then

$$\int_S f d\omega = \frac{3}{\pi} \int_E f(1, e_2, e_3) e_3^{-1/2} D^{-1/2} de_2 de_3.$$

PROOF. We may assume f is homogeneous of degree $2k$ in x (each e_i is even). By symmetry,

$$\begin{aligned} \int_{\mathbf{R}^3} f e^{-|x|^2} dx &= 48 \int_{x_1 \geq x_2 \geq x_3 \geq 0} f e^{-|x|^2} dx \\ &= 6 \int_{e_1, e_2, e_3, D \geq 0} f(e_1, e_2, e_3) \exp(-e_1) e_3^{-1/2} D^{-1/2} de_1 de_2 de_3 \end{aligned}$$

(because

$$\frac{\partial(e_1, e_2, e_3)}{\partial(x_1, x_2, x_3)} = 8x_1 x_2 x_3 \prod_{i < j} (x_i^2 - x_j^2) = 8(e_3 D)^{1/2}$$

in the specified regions). In the latter integral use the coordinates $\tilde{e}_2 = e_2/e_1^2$, $\tilde{e}_3 = e_3/e_1^3$ and the homogeneity relation $f(e_1, e_1^2 \tilde{e}_2, e_1^3 \tilde{e}_3) = e_1^k f(1, \tilde{e}_2, \tilde{e}_3)$ (as well as one for D). Then the e_1 -integral is a gamma-function which gives the value

$$\Gamma\left(k + \frac{3}{2}\right) \int_E f(1, e_2, e_3) (e_3 D)^{-1/2} de_2 de_3.$$

Combine this with Lemma 4.6 to get the result. \square

Selberg (see Askey [1] for the statement and q -analogues, Mehta [7, p. 40] for matrix ensemble applications, and Morris [8] for a study of identities related to root systems and such integrals) determined an integral from which one can derive the value of

$$\int_{\mathbf{R}^3} (x_1 x_2 x_3)^{2\alpha} \left(\prod_{i < j} (x_i^2 - x_j^2) \right)^{2\beta} e^{-|x|^2} dx.$$

Applying Lemma 4.6 to this result (Askey [1, p. 938]) and using duplication and triplication formulas for Γ , we obtain

$$\int_S e_3^\alpha D^\beta d\omega = \frac{(1/2)_\alpha (1/2)_{\alpha+\beta} (1/2)_{\alpha+2\beta} (1/2)_\beta (1/3)_\beta (2/3)_\beta}{(3/2)_{3\alpha+6\beta}} \cdot 108^\beta$$

$$(\alpha, \beta = 0, 1, 2, \dots).$$

(Thus, we do know the integral of the weight function, but as yet we do not have L^2 -norms of elements of $H_n(\alpha, \beta)$.)

Next we describe a third-order differential operator that maps $H_n(\alpha, \beta)$ to $H_{n-6}(\alpha + 2, \beta + 1)$. As some motivation for the lengthy calculation, we point out that the adjoint of ∂_2 in the case $N = 2$ is closely related to the Rodrigues formula for Jacobi polynomials (essentially $P_n^{(\alpha, \beta)}$ is the result of applying $(\partial_2^*)^n$ to $P_0^{(\alpha+n, \beta+n)}$; note each factor ∂_2^* is actually with respect to a different weight function). The structure in $N = 3$ is not as simple because $\dim H_n(\alpha, \beta)$ increases with n (roughly), nevertheless an analogue of ∂_2^* will be helpful in giving more structure to $H_n(\alpha, \beta)$. To obtain a specific complete orthogonal decomposition remains a subject for further research.

We first sketch the ideas that led to the discovery of the $N = 3$ analogue of ∂_2 . By analogy with the $N = 2$ case, for each pair (α, β) we want a differential operator T (that preserves homogeneous polynomials) and another pair (α', β') such that $(L + (\alpha' + \frac{1}{2})\delta_1 + 2\beta'\delta_2)T = T(L + (\alpha + \frac{1}{2})\delta_1 + 2\beta\delta_2)$. We write this as

$$(4.1) \quad [L, T] + (\alpha' - \alpha)\delta_1 T + 2(\beta' - \beta)\delta_2 T = -\left(\alpha + \frac{1}{2}\right)[\delta_1, T] - 2\beta[\delta_2, T].$$

If T is of order m then the left side is of order $\leq m + 1$, whereas the right side is of order m . Thus a necessary condition for such an operator is that the $(m + 1)$ st-order terms of $[L, T] + c_1\delta_1 T + c_2\delta_2 T$ vanish for some choice of c_1, c_2 . With this, one can rule out first-order operators fairly quickly.

Another important consideration is the dimension pattern mentioned above, namely, $\dim H_n(\alpha, \beta) = \dim H_{n-6}(\alpha', \beta') + 1$, so an operator that lowers the degree by 6 should have good properties. Finally, one has the feeling that ∂_1 should not appear (is e_1 not constant on E ?). Thus one arrives at trying a linear combination of $\partial_2^3, e_1\partial_2^2\partial_3, e_2\partial_2\partial_3^2, e_3\partial_3^3, \partial_3^3$.

We let

$$T = c_1\partial_2^3 + c_2e_1\partial_2^2\partial_3 + c_3e_2\partial_2\partial_3^2 + c_4e_3\partial_3^3,$$

and after lengthy calculations find that the fourth-order terms in

$$[L, T] + (c_5\partial_1 + c_6e_1\partial_2 + c_7\partial_3)T$$

are cancelled when $c_1 = c_2 = c_3 = c_4 = 1$, $c_5 = 12$, $c_6 = 6$, $c_7 = 2$ (there are 12 fourth-order terms giving 12 fortunately consistent equations for c_1 through c_7). Also $12\partial_1 + 6e_1\partial_2 + 2e_2\partial_3 = 2(\delta_1 + \delta_2)$.

4.8. DEFINITION. Let $T := \partial_2^3 + e_1\partial_2^2\partial_3 + e_2\partial_2\partial_3^2 + e_3\partial_3^3$. This is a third-order operator that lowers the degree of a polynomial by 6.

4.9. PROPOSITION. $[L, T] + 2(\delta_1 + \delta_2)T = -2[L, \partial_3^2] - 4(\delta_1 + \delta_2)\partial_3^2 = 4[\delta_2, T]$, and $[\delta_1, T] = 0$.

PROOF. We sketched the argument showing that the fourth-order terms on the left side cancel out. Calculating the remaining (all third-order) terms we obtain $4(3\partial_2 + e_1\partial_3)\partial_2\partial_3$. Further,

$$\begin{aligned}[L, \partial_3^2] &= -6\partial_2^2\partial_3 - (12\partial_1 + 8e_1\partial_2 + 2e_2\partial_3)\partial_3^2 \\ &= -2(3\partial_2 + e_1\partial_3)\partial_2\partial_3 - 2(\delta_1 + \delta_2)\partial_3^2,\end{aligned}$$

$[\delta_2, T] = (3\partial_2 + e_1\partial_3)\partial_2\partial_3$, and $[\delta_1, T] = 0$ are all straightforwardly obtained. \square

4.10. DEFINITION. Let $T_\beta := T + (\beta + 2)\partial_3^2$.

4.11. THEOREM. For each $n \geq 6$, T_β maps $H_n(\alpha, \beta)$ onto $H_{n-6}(\alpha + 2, \beta + 1)$. The kernel of T_β is spanned by

$$\sum_j \frac{(-1)^j (p - m)_{3j}}{(p + 2j)!(1 - m - \beta)_j} F_{m-j, p+2j}^{(\alpha, \beta)},$$

where $n = 2m + p$ (with $m = [n/2]$, $p = 0$ or 1) and $0 \leq j \leq (m - p)/3$.

PROOF. We find c so that

$$[L, T + c\partial_3^2] + 2(\delta_1 + \delta_2)(T + c\partial_3^2) = -(\alpha + \tfrac{1}{2})[\delta_1, T + c\partial_3^2] - 2\beta[\delta_2, T + c\partial_3^2]$$

is an identity (see (4.1)). By the relations above, this becomes $(4 - 2c)[\delta_2, T] = -2\beta[\delta_2, T]$, so that $c = \beta + 2$ is the solution (note $[\delta_1, \partial_3^2] = 0 = [\delta_2, \partial_3^2]$). Thus $L_{\alpha+2, \beta+1}T_\beta = T_\beta L_{\alpha, \beta}$, and T_β lowers the degree of a polynomial by 6.

The elements of $\partial\Omega_n$ are given by $\{(m - j, p + 2j): m = [n/2], p = n - 2m, 0 \leq j \leq (m - p)/3\}$, so suppose $T_\beta \sum_j c_j F_{m-j, p+2j}^{(\alpha, \beta)} = 0$. Since the left side is in $H_{n-6}(\alpha + 2, \beta + 1)$, it suffices to require (by Proposition 4.2) that the coefficients of $e_2^{\lambda_2 - \lambda_3} e_3^{\lambda_3}$ ($2\lambda_2 + \lambda_3 = n - 6$) be zero. Such terms can only come from $e_2^{\mu_2 - \mu_3} e_3^{\mu_3}$ ($2\mu_2 + \mu_3 = n$) (since there is no ∂_1 in T_β).

Indeed

$$\begin{aligned}T_\beta e_2^{\mu_2 - \mu_3} e_3^{\mu_3} &= (\mu_2 - \mu_3)(\mu_2 - \mu_3 - 1)(\mu_2 - \mu_3 - 2) \cdot e_2^{\mu_2 - \mu_3 - 3} e_3^{\mu_3} \\ &\quad + \mu_3(\mu_3 - 1)(\mu_2 + \beta) e_2^{\mu_2 - \mu_3} e_3^{\mu_3 - 2} \quad (+ \text{a term involving } e_1).\end{aligned}$$

Put $\mu_2 = m - j$, $\mu_3 = p + 2j$ in this relation and get another formula by replacing j by $j + 1$. The equation $T_\beta \sum_j c_j F_{m-j, p+2j}^{(\alpha, \beta)} = 0$ now gives a simple recurrence for c_j with a unique solution (with $c_0 = 1$) (the recurrence for c_{j+1} in terms of c_j is the coefficient of $e_2^{m-p-3j-3} e_3^{p+2j}$). Since $(\ker T_\beta) \cap H_n(\alpha, \beta)$ is of dimension 1, T_β must be onto $H_{n-6}(\alpha + 2, \beta + 1)$. \square

Note the difference for $N = 3$ and $N = 2$ in the contiguous (α, β) values. Observe, however, that in $N = 2$ the weight function $e_2^\alpha(e_1^2 - 4e_2)^\beta$ goes up by degree 2 in each factor, and in $N = 3$ the weight $e_3^\alpha D^\beta$ in going to $e_3^{\alpha+2} D^{\beta+1}$ has each factor's degree increased by 6.

From the general theory (Proposition 3.6) we know $H_n(\alpha, \beta) \perp H_m(\alpha, \beta)$ for $m \neq n$ in the Hilbert space $L^2(E; \alpha, \beta) := L^2(E, e_3^{\alpha-1/2} D^{\beta-1/2} de_2 de_3)$, and $L^2(E; \alpha, \beta) = \sum_{n=0}^\infty \oplus H_n(\alpha, \beta)$ (complete direct sum). Thus the operator T_β is

densely defined on $L^2(E; \alpha, \beta)$ with dense range in $L^2(E; \alpha + 2, \beta + 1)$, so it is of great interest to determine T_β^* (Rasala [10] did a detailed analysis of the role of adjoints in one-variable polynomials).

For f, g polynomials in e_2, e_3 we have the formal identity
(4.2)

$$\begin{aligned} \int_E f(T_\beta^* g) e_3^{\alpha-1/2} D^{\beta-1/2} de_2 de_3 &= \int_E (T_\beta f) g e_3^{\alpha+3/2} D^{\beta+1/2} de_2 de_3 \\ &= - \int_E f \{ (\partial_2^3 + e_3 \partial_3^3 + e_2 \partial_2 \partial_3^2 + e_1 \partial_2^2 \partial_3 + (2 - \beta)) \cdot (g e_3^{\alpha+3/2} D^{\beta+1/2}) \} de_2 de_3; \end{aligned}$$

the latter equation comes from three-fold integration by parts provided that $(g e_3^{\alpha+3/2} D^{\beta+1/2})$ vanishes of sufficiently high order on the boundary of E , so temporarily we assume $\alpha \geq 1, \beta \geq 3$. We define formally

$$T_\beta^* g := - (T_{(-\beta)} (g e_3^{\alpha+3/2} D^{\beta+1/2})) e_3^{1/2-\alpha} D^{1/2-\beta}.$$

This is indeed the adjoint of T_β for smooth g and possibly restricted α, β . However, it is not at all obvious that T_β^* maps polynomials to polynomials, although the degree of (positive) homogeneity is raised by 6. The heavy part of the following calculation is dealing with derivatives of $D^{\beta+1/2}$. Grouping by α -terms, we have

$$\begin{aligned} (4.3) \quad T_\beta^* g &= -e_3^2 (P_0 g + P_3 \partial_3 g + 2P_6 \partial_3^2 g + e_3 D \partial_3^3 g \\ &\quad + P_2 \partial_2 g + 2P_5 \partial_2 \partial_3 g + e_2 D \partial_2 \partial_3^2 g + e_1 D \partial_2^2 \partial_3 g + D \partial_2^3 g) \\ &\quad - (-\alpha - \tfrac{3}{2}) e_3 (P_3 g + 2P_6 \partial_3 g + 3e_3 D \partial_3^2 g + 2P_5 \partial_2 g + 2e_2 D \partial_2 \partial_3 g) \\ &\quad - (-\alpha - \tfrac{3}{2})_2 (P_6 g + 3e_3 D \partial_3 g + e_2 D \partial_2 g) - (-\alpha - \tfrac{3}{2})_3 D g, \end{aligned}$$

where (subscripts indicate degree)

$$\begin{aligned} P_0 &:= ((\partial_2^3 + e_1 \partial_2^2 \partial_3 + e_2 \partial_2 \partial_3^2 + e_3 \partial_3^3 + (2 - \beta) \partial_3^2) D^{\beta+1/2}) D^{1/2-\beta}, \\ P_3 &:= ((e_1 \partial_2^2 + 2e_2 \partial_2 \partial_3 + 3e_3 \partial_3^2 + 2(2 - \beta) \partial_3) D^{\beta+1/2}) D^{1/2-\beta}, \\ P_6 &:= ((e_2 \partial_2 + 3e_3 \partial_3 + (2 - \beta)) D^{\beta+1/2}) D^{1/2-\beta}, \\ P_2 &:= ((3\partial_2^2 + 2e_1 \partial_2 \partial_3 + e_2 \partial_3^2) D^{\beta+1/2}) D^{1/2-\beta}, \\ P_5 &:= ((e_1 \partial_2 + e_2 \partial_3) D^{\beta+1/2}) D^{1/2-\beta}. \end{aligned}$$

It remains to calculate the functions P_j . They are in fact polynomials. The differentiations result in polynomials (in e_1, e_2, e_3) which have to be divided by D . (The author carried out this polynomial arithmetic on his APPLE II computer.) The results are:

$$\begin{aligned} P_0 &= -6(2\beta + 1)(3\beta + 1)(3\beta + 2), \\ P_3 &= 2(2\beta + 1) \{ 6(\beta + 1)(e_1 e_2 - 9e_3) - (\beta + 2)(e_1^2 - 3e_2)e_1 \}, \\ P_6 &= 5(\beta + 1)D + 2(2\beta + 1)(e_1^2 - 3e_2)(3e_1 e_3 - e_2^2), \\ P_2 &= 6(2\beta + 1)(3\beta + 2)(e_1^2 - 3e_2), \\ P_5 &= (2\beta + 1)(9e_3 - e_1 e_2)(e_1^2 - 3e_2). \end{aligned}$$

Note that the inner bracketed expressions in e_1, e_2, e_3 all vanish at $e_1 = 1, e_2 = \frac{1}{3}, e_3 = \frac{1}{27}$ (the cusp).

4.12. THEOREM. *The adjoint of T_β on polynomials in $L^2(E; \alpha + 2, \beta + 1)$ is given by (4.3) and maps $H_n(\alpha + 2, \beta + 1)$ into $H_{n+6}(\alpha, \beta)$. This holds for all $\alpha, \beta \geq 0$.*

PROOF. Let $g \in H_n(\alpha + 2, \beta + 1)$ and let f be a polynomial of degree $< n + 6$. By definition,

$$\int_E (T_\beta^* g) f e_3^{\alpha-1/2} D^{\beta-1/2} de_2 de_3 = \int_E g (T_\beta f) e_3^{\alpha+3/2} D^{\beta+1/2} de_2 de_3.$$

But $T_\beta f$ is a polynomial of degree $< n$, so, by the orthogonal decomposition of $L^2(E; \alpha + 2, \beta + 1)$, the latter integral is zero. Thus $T_\beta^* g$ is a homogeneous polynomial of degree $n + 6$ which is orthogonal to all lower-degree polynomials in $L^2(E; \alpha, \beta)$, hence $T_\beta^* g \in H_{n+6}(\alpha, \beta)$. The identity (4.2) holds for any two polynomials f, g for infinitely many values of α, β so it will hold for all α, β for which both sides are finite (this includes $\alpha, \beta \geq 0$). \square

4.13. COROLLARY. *For $n \geq 6$, $T_\beta^* H_{n-6}(\alpha + 2, \beta + 1)$ is contained in the subspace of $H_n(\alpha, \beta)$ of functions that vanish at $(0, 0) \in E$; that is, they have no e_1^n term.*

PROOF. A detailed examination of $T_\beta^* g$ in (4.3) shows that every term contains one of the following as a multiplier: $e_3, D, 3e_1 e_3 - e_2^2$ (and D itself has no e_1^6 term). Each of these vanishes at $(0, 0)$. \square

Of course we now have an orthogonal decomposition of $H_n(\alpha, \beta)$ as $\ker T_\beta \oplus T_\beta^* H_{n-6}(\alpha + 2, \beta + 1)$. One would like an explicit orthogonal decomposition of $H_n(\alpha, \beta)$ into one-dimensional subspaces. Possibly one can explicitly find the eigenvectors of $T_\beta^* T_\beta$. The computations for this at least can be restricted to considering the coefficients of $\phi(\lambda)$ with $\lambda_1 = \lambda_2$. One also desires L^2 -norms and special values (at $(0, 0), (\frac{1}{4}, 0), (\frac{1}{3}, \frac{1}{27})$) of such eigenvectors. Besides these basic problems there are, as usual, the questions about relations among polynomials of different (α, β) families and known families of one- or two-variable polynomials.

It does seem that the polynomials of this paper are a quite natural analogue of the Jacobi polynomials, unifying the fundamental concepts of reflection groups, harmonic functions, and multivariate beta distributions.

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF VIRGINIA, CHARLOTTESVILLE, VIRGINIA 22903